PHY 711: ANALYTICAL DYNAMICS

Additional Practice Problems III

Problem 1

The velocity of a particle $\frac{dx^i}{dt}$ does not have a nice transformation property since both x^i and t change and get mixed under Lorentz transformations. For this reason, one defines a 4-velocity u^μ which transforms as a 4-vector under Lorentz transformations. The actual 3-vector v^i corresponding to the velocity can then be defined by

$$\frac{v^i}{c} = \frac{u^i}{u^0}$$

a) Consider a particle moving aong the x_1-direction, so that $u^\mu = (u^0, u^1, 0, 0)$. Consider the velocity of the particle in a frame moving with velocity $w^i = (w^1, 0, 0)$. This can be done by carrying out a Lorentz transformation and then forming the ratio as above. In this way, obtain the relativistic law for addition of velocities along the same direction.
b) Consider the same set-up but with the frame moving along the x_2-direction, so that $w^i = (0, w^2, 0)$. What is the addition law for velocities now?

Solution

a) Consider the Lorentz transformation of the 4-vector $(u^0, u^1, 0, 0)$ with parameter w^1/c,

$$u'^0 = \gamma(u^0 - w^1u^1/c), \quad u'^1 = \gamma(u^1 - w^1u^0/c), \quad u'^2 = u^2 = 0, \quad u'^3 = u^3 = 0$$

where $\gamma^{-1} = \sqrt{1 - w^2/c^2}$. The transformed velocity is given by

$$V^1 = c\frac{u'^1}{u'^0} = c\frac{u^1 - w^1u^0/c}{u^0 - w^1u^1/c} = \frac{v^1 - w^1}{1 - w^1v^1/c^2}$$

b) The Lorentz transformation now becomes

$$\begin{aligned} u'^0 &= \gamma(u^0 - w^2u^2/c) = \gamma u^0 \\ u'^1 &= u^1, \qquad u'^3 = u^3 \\ u'^2 &= \gamma(u^2 - w^2u^0/c) = -\gamma w^2 u^0/c \end{aligned}$$

The velocity now has components both along the x_1 and x_2 directions.

$$V^i = v^1\sqrt{1 - (w^2)^2/c^2}, \qquad V^2 = -w^2, \qquad V^3 = 0$$

This shows that, again, the nonrelativistic vectorial addition of velocities does not hold in the fully relativistic theory.

Problem 2

A particle moves in one dimension x under the influence of a potential

$$V(x) = -b\frac{1}{\cosh^2 ax}$$

where a, b are positive constants. Use the Hamilton-Jacobi method to find the solution (trajectory of the particle) for negative values of the energy. (This potential is important in the context of soliton solutions for the Korteweg-de Vries equation for surfaces waves in a shallow body of water.)

Solution

The Hamiltonian is evidently given by

$$H = \frac{p^2}{2m} - b\frac{1}{\cosh^2 ax}$$

The Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - b\frac{1}{\cosh^2 ax} = 0$$

Writing $S = W - Et$, we find

$$W' = \pm\sqrt{2mE + \frac{2mb}{\cosh^2 ax}} \implies W = \pm\int dx\,\sqrt{2mE + \frac{2mb}{\cosh^2 ax}}$$

The only constant of integration is E, so setting the derivative of S with respect to E to a constant $-t_0$, we find

$$t - t_0 = \pm m\int \frac{dx}{\sqrt{2mE + \frac{2mb}{\cosh^2 ax}}}$$

The potential is negative, symmetric around the origin and tapers off to zero as $|x| \to \infty$. The minimum is at $x = 0$ where $V = -b$. Thus we need $E > -b$. We choose negative values of E, say, $E = -\epsilon$, $\epsilon > 0$. We can the write

$$\begin{aligned} t - t_0 &= \pm\sqrt{\frac{m}{2\epsilon}}\int \frac{dx\,\cosh ax}{\sqrt{\frac{(b-\epsilon)}{\epsilon} - \sinh^2 ax}} \\ &= \pm\sqrt{\frac{m}{2a^2\epsilon}}\int \frac{du}{\sqrt{\frac{(b-\epsilon)}{\epsilon} - u^2}}, \qquad u = \sinh ax \\ &= \pm\sqrt{\frac{m}{2a^2\epsilon}}\sin^{-1}\left(u\sqrt{\frac{\epsilon}{b-\epsilon}}\right) \end{aligned}$$

Notice that we do have $b - \epsilon > 0$. Inverting this equation we find

$$x(t) = \frac{1}{a} \sinh^{-1} \left[\sqrt{\frac{b-\epsilon}{\epsilon}} \sin \left(\sqrt{\frac{2\epsilon a^2}{m}} (t - t_0) \right) \right]$$

The initial condition is that at $t = t_0$, the particle is at the center $x = 0$.

Problem 3

The Lagrangian for a spinning top on the floor (i.e., subject to gravity) is given by

$$\mathcal{L} = \frac{1}{2} I_1 (\dot\theta^2 + \sin^2\theta\, \dot\psi^2) + \frac{1}{2} I_3 (\dot\varphi + \dot\psi \cos\theta)^2 - Mgh\, \cos\theta$$

(This is what is called Lagrange top; we discussed this in class.)

a) Identify the canonical momenta and the Hamiltonian.

b) Write down the Hamilton-Jacobi equation and find the general solution.

Solution

The canonical momenta are given by

$$\begin{aligned} p_\theta &= \frac{\partial L}{\partial \dot\theta} = I_1 \dot\theta \\ p_\psi &= \frac{\partial L}{\partial \dot\psi} = I_1 \sin^2\theta\, \dot\psi + I_3 \cos\theta (\dot\varphi + \dot\psi \cos\theta) \\ p_\varphi &= \frac{\partial L}{\partial \dot\varphi} = I_3 (\dot\varphi + \dot\psi \cos\theta) \end{aligned}$$

Using the last of these, the first two can be solved for the velocities as

$$\dot\theta = \frac{p_\theta}{I_1}, \qquad \dot\psi = \frac{p_\psi - \cos\theta\, p_\varphi}{I_1 \sin^2\theta}, \qquad \dot\varphi = \frac{p_\varphi}{I_3} - \cos\theta\, \dot\psi = \frac{p_\varphi}{I_3} - \cos\theta \left[\frac{p_\psi - \cos\theta\, p_\varphi}{I_1 \sin^2\theta} \right]$$

The Hamiltonian is obtained as

$$\begin{aligned} H &= p_\theta \dot\theta + p_\psi \dot\psi + p_\varphi \dot\varphi - L \\ &= p_\theta \frac{p_\theta}{I_1} - \tfrac{1}{2} I_1 \left(\frac{p_\theta}{I_1} \right)^2 + p_\varphi \left(\frac{p_\varphi}{I_3} - \dot\psi \cos\theta \right) - \tfrac{1}{2} I_3 \left(\frac{p_\varphi}{I_3} \right)^2 + p_\psi \dot\psi - \tfrac{1}{2} I_1 \sin^2\theta\, \dot\psi^2 + Mgh \cos\theta \\ &= \frac{p_\theta^2}{2I_1} + \frac{p_\varphi^2}{2I_3} + (p_\psi - p_\varphi\, \cos\theta) \dot\psi - \tfrac{1}{2} I_1 \sin^2\theta\, \dot\psi^2 + Mgh \cos\theta \\ &= \frac{p_\theta^2}{2I_1} + \frac{p_\varphi^2}{2I_3} + \frac{(p_\psi - p_\varphi\, \cos\theta)^2}{2I_1 \sin^2\theta} + Mgh \cos\theta \end{aligned}$$

The Hamilton-Jacobi equation takes the form

$$\frac{\partial S}{\partial t} + \frac{1}{2I_1} \left(\frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{2I_3} \left(\frac{\partial S}{\partial \varphi} \right)^2 + \frac{1}{2I_1 \sin^2\theta} \left(\frac{\partial S}{\partial \psi} - \frac{\partial S}{\partial \varphi} \cos\theta \right)^2 + Mgh \cos\theta = 0$$

Since the various coefficients are independent of t, φ, ψ, we can separate variables as

$$S = -Et + l_\varphi \varphi + l_\psi \psi + W(\theta)$$

The Hamilton-Jacobi equation reduces to the form

$$\begin{aligned}
\frac{1}{2I_1}\left(\frac{\partial W}{\partial \theta}\right)^2 &= \mathcal{E} - V_{\rm eff} \\
V_{\rm eff} &= \frac{1}{2I_1 \sin^2\theta}\,(l_\psi - l_\varphi \cos\theta)^2 + Mgh\cos\theta \\
\mathcal{E} &= E - \frac{l_\varphi^2}{2I_3}
\end{aligned}$$

The solution can be expressed as the integral

$$S = -Et + l_\varphi \varphi + l_\psi \psi + \int d\theta \, \sqrt{2I_1(\mathcal{E} - V_{\rm eff})}$$

This is the general solution for S. By taking derivatives with respect to E, l_φ, l_ψ ansd setting them to constants, we find the solution for the angles as

$$\begin{aligned}
t - t_0 &= \int d\theta \sqrt{\frac{I_1}{2(\mathcal{E} - V_{\rm eff})}} \\
-(\varphi - \varphi_0) &= \int d\theta \sqrt{\frac{I_1}{2(\mathcal{E} - V_{\rm eff})}} \left[-\frac{l_\varphi}{I_3} - \frac{1}{I_1 \sin^2\theta} \cos\theta \, (l_\psi - l_\varphi \cos\theta) \right] \\
-(\psi - \psi_0) &= \int d\theta \sqrt{\frac{I_1}{2(\mathcal{E} - V_{\rm eff})}} \left[\frac{(l_\psi - l_\varphi \cos\theta)}{I_1 \sin^2\theta} \right]
\end{aligned}$$

These equations give the complete solution. The explicit evaluation of the integrals will require elliptic functions.

Problem 4

The Lagrangian for a nonrelativistic particle interacting with an external magnetic field is given by

$$\mathcal{L} = \tfrac{1}{2}\, m \dot{x}_i \dot{x}_i + e A_i \dot{x}_i$$

where $i = 1, 2, 3$ and there is summation over the index i.

a) Obtain the Hamiltonian in terms of the canonical momenta and coordinates.

b) Now consider a uniform magnetic field in the the z direction with $A_i = -\frac{1}{2} eB \epsilon_{ij} x^j$, for $i, j = 1, 2$, $A_3 = 0$. Ignore motion in the z-direction in what follows. Define a change of variables to

$$Q_1 = \frac{1}{\sqrt{eB}}\,(p_1 + \tfrac{1}{2} eB x_2)$$

$$
\begin{aligned}
Q_2 &= \frac{1}{\sqrt{eB}}\left(p_2 + \tfrac{1}{2}eBx_1\right) \\
P_1 &= \frac{1}{\sqrt{eB}}\left(p_2 - \tfrac{1}{2}eBx_1\right) \\
P_2 &= \frac{1}{\sqrt{eB}}\left(p_1 - \tfrac{1}{2}eBx_2\right)
\end{aligned}
$$

Obtain the Poisson brackets $\{Q_i, Q_j\}$, $\{P_i, P_j\}$, $\{Q_i, P_j\}$.
c) Write the Hamiltonian in terms of these new variables. Obtain the equations of motion for all Q_i, P_i, $i = 1, 2$.
d) What are the conserved quantities? Solve the equations of motion to obtain the general solution.

<u>Solution</u>
a) The canonical momenta are given by

$$
p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + eA_i \implies \dot{x}_i = \frac{(p - eA)_i}{m}
$$

The Hamiltonian is thus

$$
\begin{aligned}
H &= p_i(\dot{x}_i - L = p_i \frac{(p - eA)_i}{m} - \tfrac{1}{2}m\left(\frac{(p - eA)_i}{m}\right)^2 - eA_i \frac{(p - eA)_i}{m} \\
&= \frac{(p - eA)^2}{2m}
\end{aligned}
$$

This is the Hamiltonian for a charged particle (of charge e) in a magnetic field.
b) For the given uniform magnetic field we have $A_1 = -\frac{1}{2}Bx_2$, $A_2 = \frac{1}{2}Bx_1$, $A_3 = 0$. (There is an extra e in the question, it should not be there. If you keep it, you will get essentially the same answer except for an extra e.) For the Poisson brackets, we can use linearity, antisymmetry of the brackets and the basic bracket relations

$$
\{x_i, x_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0
$$

Since $\{Q_1, Q_1\} = \{Q_2, Q_2\} = 0$ by antisymmetry, we only need $\{Q_1, Q_2\}$. It is given by

$$
\{Q_1, Q_2\} = \frac{1}{eB}\{p_1 + \tfrac{1}{2}eBx_2, p_2 + \tfrac{1}{2}eBx_1\} = \frac{1}{2}\left(\{p_1, x_1\} + \{x_2, p_2\}\right) = \frac{1}{2}(1 - 1) = 0
$$

In a similar way

$$
\begin{aligned}
\{P_1, P_2\} &= -\frac{1}{2}\left(\{p_2, x_2\} + \{x_1, p_1\}\right) = 0 \\
\{Q_1, P_1\} &= \frac{1}{2}\left(-\{p_1, x_1\} + \{x_2, p_2\}\right) = 1
\end{aligned}
$$

$$
\begin{aligned}
\{Q_2, P_2\} &= \frac{1}{2}\left(-\{p_2, x_2\} + \{x_1, p_1\}\right) = 1 \\
\{Q_1, P_2\} &= \{Q_2, P_1\} = 0
\end{aligned}
$$

In the last two, we have only dissimilar p_i, x_j, so they vanish. We have two sets of canonical variables (Q_1, P_1) and (Q_2, P_2) with the first set having zero Poisson brackets with the second.

c) The Hamiltonian can be written as

$$
H = \frac{1}{2m}\left[(p_1 + \tfrac{1}{2}eBx_2)^2 + (p_2 - \tfrac{1}{2}eBx_1)^2\right] = \tfrac{1}{2}\omega\left(P_1^2 + Q_1^2\right), \quad \omega = \frac{eB}{m}
$$

This is essentially the Hamiltonian for a harmonic oscillator. The equations of motion are given by

$$
\begin{aligned}
\dot{Q}_1 &= \{Q_1, H\} = \omega\, P_1, \quad \dot{P}_1 = \{P_1, H\} = -\omega\, Q_1 \\
\dot{Q}_2 &= \{Q_2, H\} = 0, \quad \dot{P}_2 = \{P_2, H\} = 0
\end{aligned}
$$

d) Evidently, Q_2, P_2 are constants of motion, in addition to the Hamiltonian itself. The solution for these is given by

$$
Q_2 = D_2, \quad P_2 = D_1
$$

The equations for Q_1, P_1 look like the derivatives of sine and cosine functions, so the solution is trivially obtained as

$$
Q_1 = \frac{C}{\sqrt{eB}}\sin(\omega t + \varphi), \quad P_1 = \frac{C}{\sqrt{eB}}\cos(\omega t + \varphi)
$$

C and φ are arbitrary constants (related to initial values). We could absorb $\sqrt{eB}$ into C but we keep it like this for simplification later. We can no solve for the original x_i, p_i using the definition of these variables as

$$
\begin{aligned}
p_1 &= D_1 + C\sin(\omega t + \varphi), \qquad p_2 = D_2 + C\cos(\omega t + \varphi) \\
x_1 &= \frac{1}{eB}\left[D_2 - C\cos(\omega t + \varphi)\right], \quad x_2 = \frac{1}{eB}\left[-D_1 + C\sin(\omega t + \varphi)\right]
\end{aligned}
$$

With 4 arbitrary constants, combinations of which will serve as the initial values for the four variables x_i, p_i, $i = 1, 2$, we have a complete solution.

Problem 5

A charged particle which moves in a medium (with refractive index n) with a speed v which is greater than the speed of light in the medium (c/n) will emit radiation, known as the

Cerenkov radiation. Consider Cerenkov radiation from an electron in such a medium; this may be considered as the emission of a photon, namely, the process $e^- \to e^- + \gamma$. The four-vector of energy and momentum for the emitted photon may be taken as $(\hbar\omega/c, \hbar\vec{k})$ with $\vec{k}\cdot\vec{k}/n^2 = \omega^2/c^2$. (Here $\hbar$ is Planck's constant.) Show that the radiation is in a cone and obtain the opening angle of the cone.

<u>**Solution**</u>

This is a simple photon emission process, so we must have conservation of energy and momentum. Let p^μ and p'^μ denote the 4-momentum of the electron before and after the emission of the photon. We then have the condition

$$p^\mu = p'^\mu + K^\mu, \qquad K^\mu = (\hbar\omega/c, \hbar\vec{k})$$

There are four equations here, for $\mu = 0, 1, , 23$, corresponding to the conservation of energy ($\mu = 0$) and the three components of momentum. The key results we have for the electron momenta are

$$\eta_{\mu\nu} p^\mu p^\nu = \eta_{\mu\nu} p'^\mu p'^\nu = m^2 c^2$$

We can utilize this by taking p'^μ to the left hand side and taking the appropriate square with $\eta_{\mu\nu}$. This gives

$$\eta_{\mu\nu}(p^\mu - K^\mu)(p^\nu - K^\nu) = \eta_{\mu\nu} p'^\mu p'^\nu = m^2 c^2$$

Opening up the brackets and simplifying we get

$$-2\eta_{\mu\nu} p^\mu K^\nu + \eta_{\mu\nu} K^\mu K^\nu = 0$$

Using the relation $n^2\omega^2/c^2 = \vec{k}\cdot\vec{k}$, this becomes

$$\frac{E\omega}{c^2} - pn\frac{\omega}{c}\cos\theta - \frac{1}{2c^2}\hbar\omega^2(1 - n^2) = 0$$

where we used $\vec{p}\cdot\vec{k} = pk\cos\theta$, θ being the angle at which the photon is emitted as measured from the initial momentum of the electron. In the classical limit, the last term involving $\hbar$ is negligible and we find

$$\cos\theta \approx \frac{c}{vn}, \qquad v = \frac{c^2 p}{E}$$

v is the speed of the particle, as verified from

$$v_i = \frac{\partial H}{\partial p_i}, \qquad H = \sqrt{c^2p^2 + m^2c^4}$$

The equation for θ gives the opening angle of the cone as $\theta = \cos^{-1}(c/vn)$. For a real angle, we need $\cos\theta < 1$. This requires $v > c/n$. Thus the speed of the particle must exceed the speed of light in the medium for Cerenkov radiation to occur.

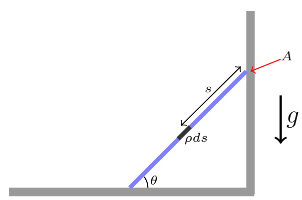
Problem 6

A rod of very small thickness rests against a wall as shown. The rod can slide down, with point of contact (shown as A) always touching the wall.

a) Obtain the Lagrangian describing this motion. (Hint: *Consider a small mass element of mass ρds at a distance s from the point A, as shown. Obtain the coordinates of this element to construct the kinetic energy for this element and then integrate over the length of the rod.*)

b) Identify the canonical momentum and the Hamiltonian.

c) Write down the Hamilton-Jacobi equation and its solution as an integral over θ. (The integral can be done in terms of elliptic functions, but you do not have to do that.)



Problem 6

Solution

a) The indicated mass element has a mass $= \rho ds$, where ρ is the mass per unit length. The Cartesian coordinates of this mass element are

$$x = -s\,\cos\theta, \qquad y = L\sin\theta - s\,\sin\theta$$

where L is the length of the rod. The kinetic energy is thus given by

$$T = \frac{1}{2}\rho ds\left[(s\sin\theta\dot{\theta})^2 + ((L-s)\cos\theta\dot{\theta})^2\right]$$

Integrating this over s from zero to L and using

$$\int_0^L ds\, s^2 = \int_0^L ds\,(L-s)^2 = \rho\frac{L^3}{3} = \frac{1}{3}ML^2,$$

we find the total kinetic energy as

$$T = \frac{1}{6}ML^2\dot{\theta}^2$$

The position of the center of mass, which is also the geometric center for uniform density for the rod, is given by

$$X = -\frac{L}{2}\cos\theta, \qquad Y = \frac{L}{2}\sin\theta$$

The potential energy is thus

$$V = \frac{MgL}{2}\sin\theta$$

giving

$$L = \frac{1}{6}ML^2\dot{\theta}^2 - \frac{MgL}{2}\sin\theta$$

b) There is only one generalized coordinate, θ. Thus

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = \frac{1}{3}ML^2\,\dot{\theta} \implies \dot{\theta} = \frac{3p_\theta}{ML^2}$$

$$H = p_\theta\dot{\theta} - L = \frac{3}{2ML^2}p_\theta^2 + \frac{MgL}{2}\sin\theta$$

c) The Hamilton-Jacobi equation takes the form

$$\frac{\partial S}{\partial t} + \frac{3}{2ML^2}\left(\frac{\partial S}{\partial\theta}\right)^2 + \frac{MgL}{2}\sin\theta = 0$$

Separating variables as $S = -Et + W$, we find

$$S = -Et + \int d\theta\,\sqrt{\frac{2ML^2E}{3} - g\frac{M^2L^3}{3}\sin\theta}$$

Taking the derivative with respect to the constant of integration E and setting it to a constant (-t_0), we get

$$t - t_0 = \frac{ML^2}{3}\int d\theta\,\left]\frac{2ML^2E}{3} - g\frac{M^2L^3}{3}\sin\theta\right]^{-\frac{1}{2}}$$

This gives the general solution.

Problem 7

A particle of mass m is constrained to move on a circle of radius R in the (x, y)-plane, $x^2 + y^2 = R^2$. There is a potential energy for the motion given by $V = \frac{1}{2}(ax^2 + by^2)$, where a, b are positive constants with $a > b$.

a) Write down the Lagrangian and the Hamiltonian.

b) Identify the equilibrium points and determine which are the stable equilibrium points.

c) Determine the frequency of small oscillations around the stable equilibrium points.

Solution

Since the particle is constrained to move in a circle, the radius does not change with time. So we write the Cartesian coordinates as

$$x = R\cos\varphi, \qquad y = R\sin\varphi$$

This ensures $x^2 + y^2 = R^2$. The kinetic energy of the particle is thus

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}mR^2\dot{\varphi}^2$$

The potential energy is given by

$$\begin{aligned} V &= \frac{1}{2}\left(aR^2\cos^{\varphi} + bR^2\sin^2\varphi\right) = \frac{1}{2}R^2(a-b)\cos^2\varphi + \frac{1}{2}bR^2 \\ &= \frac{1}{4}R^2(a-b)\cos(2\varphi) + \frac{a+b}{4}R^2 \end{aligned}$$

The Lagrangian is

$$L = \frac{1}{2}mR^2\dot{\varphi}^2 - \frac{1}{4}R^2(a-b)\cos(2\varphi) + \text{constant}$$

Further

$$\begin{aligned} p_\varphi &= \frac{\partial L}{\partial\dot{\varphi}} = mR^2\dot{\varphi} \implies \dot{\varphi} = \frac{p_\varphi}{mR^2} \\ H &= p_\varphi\dot{\varphi} - L = \frac{p_\varphi^2}{2mR^2} + \frac{1}{4}R^2(a-b)\cos(2\varphi) + \text{constant} \end{aligned}$$

The Lagrangian equation of motion becomes

$$mR^2\ddot{\varphi} = \frac{R^2}{2}(a-b)\sin(2\varphi)$$

The equilibrium points correspond to $\ddot{\varphi} = 0$, i.e., they are given by $\sin(2\varphi) = 0$. The solutions are

$$\varphi = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2} \equiv \varphi_0$$

For a small perturbation $\varphi = \varphi_0 + \eta$ around these points, the equation of motion becomes

$$mR^2\ddot{\eta} = \frac{R^2}{2}(a-b)\sin(2\varphi_0 + 2\eta) \approx R^2(a-b)\cos(2\varphi_0)\,\eta + \cdots$$

For stable points, the coefficient of η on the right hand side must be negative. Since $a-b > 0$, we see that $\varphi_0 = \pi/2, 3\pi/2$ are stable, while $\varphi_0 = 0, \pi$ are unstable equilibrium points. For the stable points, we find

$$\ddot{\eta} \approx -\frac{(a-b)}{m}\,\eta$$

giving the frequency as $\omega = \sqrt{(a-b)/m}$.

Problem 8

In quantum mechanics, one is familiar with the step-up and step-down operators $a^\dagger$ and a. Here we will consider a classical analog. The Hamiltonian is given by

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$

Define the complex combinations

$$\frac{p}{\sqrt{2m\omega}} + i\sqrt{\frac{m\omega}{2}}\, q = a_{\rm cl} = \sqrt{I}\, e^{i\varphi}, \qquad \frac{p}{\sqrt{2m\omega}} - i\sqrt{\frac{m\omega}{2}}\, q = a^*_{\rm cl} = \sqrt{I}\, e^{-i\varphi}$$

a)Write down the Hamiltonian in terms of the new variables I and φ.
b) Show that the change of variables from (p, q) to (I, φ) is canonical.

Solution

The given formulae can be solved for p, q as

$$p = \sqrt{2m\omega}\,\sqrt{I}\cos\varphi, \qquad q = \sqrt{\frac{2}{m\omega}}\,\sqrt{I}\sin\varphi$$

a) Using these in the expression for the Hamiltonian, we get

$$H = \omega\, I$$

b) Consider a PB in terms of I, φ as

$$\{f, g\} = \frac{\partial f}{\partial\varphi}\frac{\partial g}{\partial I} - \frac{\partial f}{\partial I}\frac{\partial g}{\partial\varphi}$$

By using the expressions for p, q and the chain rule

$$\begin{aligned}
\frac{\partial f}{\partial\varphi} &= \frac{\partial f}{\partial p}\frac{\partial p}{\partial\varphi} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial\varphi} = -\sqrt{2m\omega}\sqrt{I}\sin\varphi\frac{\partial f}{\partial p} + \sqrt{\frac{2}{m\omega}}\sqrt{I}\cos\varphi\frac{\partial f}{\partial q}\\
\frac{\partial g}{\partial I} &= \frac{\partial g}{\partial p}\frac{\partial p}{\partial I} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial I} = \sqrt{m\omega/2}\frac{1}{\sqrt{I}}\cos\varphi\frac{\partial g}{\partial p} + \frac{1}{\sqrt{2m\omega}}\frac{1}{\sqrt{I}}\sin\varphi\frac{\partial g}{\partial q}
\end{aligned}$$

Thus

$$\begin{aligned}
\{f, g\}_{I,\varphi} = \frac{\partial f}{\partial\varphi}\frac{\partial g}{\partial I} - \frac{\partial f}{\partial I}\frac{\partial g}{\partial\varphi} &= \cos^2\varphi\,\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \sin^2\varphi\,\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - (f\leftrightarrow g)\\
&= \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} = \{f, g\}_{p,q}
\end{aligned}$$

This shows that the PBs are unchanged and hence the transformation is canonical.